A model structure on categories related to categories of complexes

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Abstract

We prove a theorem of Hinich type on existence of a model structure on a category related by an adjunction to the category of differential graded modules over a graded commutative ring.

1. Introduction

Hinich proved in [Hin97] a theorem on existence of a model structure on a category related by an adjunction to the category of complexes. In this article we give a detailed proof of a theorem of similar kind. The two theorems differ at least in two points. First, Hinich works with **dg**-modules over a (commutative) ring, and we consider differential graded modules over a *graded* commutative ring $\Bbbk$. Second, in the proof Hinich introduces certain morphisms which he calls elementary trivial cofibrations and shows that any trivial cofibration is a retract of countable composition of elementary ones. We show that a trivial cofibration is a retract of an elementary trivial cofibration in our sense.

We apply our theorem to proving that categories of bi- or poly-modules over non-symmetric operads have a model structure [Lyu11, Lyu12]. For modules over operads a model structure was constructed by Harper [Har10, Theorem 1.7]. Since Hinich's article [Hin97] a plenty of results appeared in which given a (monoidal) model category one produces a model structure on another category related to the first category by an adjunction [BM03, Section 2.5], on category of monoids [SS00, Theorem 3.1] or on the category of operads [Spi01, Remark 2], [Mur11, Theorem 1.1]. Clearly, in this approach one must have a model category to begin with. The category of differential (unbounded) graded $\Bbbk^0$-modules has a projective model structure for a commutative ring $\Bbbk^0$ [CH02]. The same result for *graded commutative* ring $\Bbbk$ has to be deduced from the case of commutative ring $\Bbbk^0$ along the lines of [BM03]. After that one has to prove that $\mathbf{dg}$-$\Bbbk$-mod is a monoidal model category, which requires detailed information on cofibrations. Such information is provided e.g. by the proof of Hinich type theorem: any cofibration is a retract of a countable composition of elementary cofibrations (of a concrete form). Thus, a technical

work does not seem to be avoidable in any approach. One more reason to follow the Hinich's approach is pedagogical: it can be explained to students in detail as well as in examples.

1.1. Notations and conventions. In this article 'graded' means $\mathbb{Z}$-graded. Let $\Bbbk$ be a graded commutative ring (equipped with zero differential). By $\mathbf{gr} = \mathbf{gr}_\Bbbk = \mathbf{gr}\text{-}\Bbbk\text{-mod}$ we denote the closed category of $\mathbb{Z}$-graded $\Bbbk$-modules with $\Bbbk$-linear homomorphisms of degree 0. Thus an object of $\mathbf{gr}$ is $X = (X^m)^{m\in\mathbb{Z}}$. Symmetry in the monoidal category of graded $\Bbbk$-modules is chosen as $c(x \otimes y) = (-1)^{ml} y \otimes x$ for $x \in X^m$, $y \in Y^l$.

The abelian category $\mathbf{dg} = \mathbf{dg}\text{-}\Bbbk\text{-mod}$ is the closed category of differential $\mathbb{Z}$-graded $\Bbbk$-modules with chain $\Bbbk$-linear homomorphisms. Monomorphisms and epimorphisms of $\mathbf{dg}$ are componentwise injections and surjections. A quasi-isomorphism $M \to N \in \mathbf{dg}$ is a chain $\Bbbk$-linear homomorphism inducing an isomorphism in homology. For $a \in \mathbb{Z}$ the shift functor is defined by $[a] : \mathbf{dg} \to \mathbf{dg}$, $M \mapsto M[a]$, $M[a]^z = M^{z+a}$. The shift functor extends componentwise to $\mathbf{dg}^S$ for any set S.

Denote by $\sigma^a : M \to M[a]$ the "identity map" of degree $\deg \sigma^a = -a$. Write elements of $M[a]$ as $m\sigma^a$. When $f : V \to X$ is a homogeneous map of certain degree, the map $f[a] : V[a] \to X[a]$ is defined as $f[a] = (-1)^{fa} \sigma^{-a} f \sigma^a$. In particular, the differential $d : M \to M$ of degree 1 in a $\mathbf{dg}$-module M induces the differential $d[a] = (-1)^a \sigma^{-a} d \sigma^a : M[a] \to M[a]$ in $M[a]$. The degree 0 isomorphisms $\sigma^{-a} \cdot (\sigma^a \otimes 1) : (V \otimes W)[a] \to (V[a]) \otimes W$, $(v \otimes w)\sigma^a \mapsto (-1)^{wa} v\sigma^a \otimes w$, and $\sigma^{-a} \cdot (1 \otimes \sigma^a) : (V \otimes W)[a] \to V \otimes (W[a])$, $(v \otimes w)\sigma^a \mapsto v \otimes w\sigma^a$, are graded natural. This means that for arbitrary homogeneous maps $f : V \to X$, $g : W \to Y$ the following squares commute:

$$\begin{tikzcd}
(V[a]) \otimes W \arrow[d, "(f[a])\otimes g"'] & (V \otimes W)[a] \arrow[l, "\sigma^{-a}\cdot(\sigma^a\otimes 1)"', "\sim"] \arrow[r, "\sigma^{-a}\cdot(1\otimes\sigma^a)", "\sim"'] \arrow[d, "(f\otimes g)[a]"'] & V \otimes (W[a]) \arrow[d, "f\otimes(g[a])"] \\
(X[a]) \otimes Y & (X \otimes Y)[a] \arrow[l, "\sigma^{-a}\cdot(\sigma^a\otimes 1)"', "\sim"] \arrow[r, "\sigma^{-a}\cdot(1\otimes\sigma^a)", "\sim"'] & X \otimes (Y[a])
\end{tikzcd}$$

Actually, the second isomorphism is "more natural" than the first one, not only because it does not have a sign, but also because it suits better the right operator system of notations, accepted in this paper. In the following we always identify $(V \otimes W)[a]$ with $V \otimes (W[a])$ via $\sigma^{-a} \cdot (1 \otimes \sigma^a)$.

Assume that $\alpha : M \to N \in \mathbf{dg}$. Denote by $\operatorname{Cone}\alpha = (M[1] \oplus N, d_{\mathrm{Cone}}) \in \operatorname{Ob}\mathbf{dg}$ the graded $\Bbbk$-module with the differential

$$d_{\mathrm{Cone}} = \begin{pmatrix} d_M[1] & \sigma^{-1}\alpha \\ 0 & d_N \end{pmatrix} = \begin{pmatrix} -\sigma^{-1} d_M \sigma & \sigma^{-1}\alpha \\ 0 & d_N \end{pmatrix}.$$

The following result generalizes a theorem of Hinich [Hin97, Section 2.2].

1.2 Theorem. *Suppose that S is a set, a category $\mathcal{C}$ is complete and cocomplete and $F : \mathbf{dg}^S \rightleftarrows \mathcal{C} : U$ is an adjunction. Assume that U preserves filtering colimits. For*

any $x \in S$ consider the object $\mathbb{K}_x$ of $\mathbf{dg}^S$, $\mathbb{K}_x(x) = \mathrm{Cone}(\mathrm{id}_{\Bbbk})$, $\mathbb{K}_x(y) = 0$ for $y \ne x$. Assume that the chain map $U(\mathrm{in}_2) : UA \to U(F(\mathbb{K}_x[p]) \sqcup A)$ is a quasi-isomorphism for all objects A of $\mathcal{C}$ and all $x \in S$, $p \in \mathbb{Z}$. Equip $\mathcal{C}$ with the classes of weak equivalences (resp. fibrations) consisting of morphisms f of $\mathcal{C}$ such that Uf is a quasi-isomorphism (resp. an epimorphism). Then the category $\mathcal{C}$ is a model category.

2. Proof of existence of model structure

This section is devoted to proof of Theorem 1.2, whose hypotheses we now assume. The proof follows that of Hinich's theorem [Hin97, Section 2.2] ideologically but not in details. Constructions used in the proof describe cofibrations and trivial cofibrations in $\mathcal{C}$.

Denote the functor U also by $-^\#$, $UX = X^\#$ for $X \in \mathrm{Ob}\,\mathcal{C}$ or $X \in \mathrm{Mor}\,\mathcal{C}$. Let $\varepsilon : FUA \to A$ be the adjunction counit and let $\eta : M \to UFM$ be the adjunction unit. The adjunction bijection is given by mutually inverse maps

$$(l : FM \to A) \longmapsto l^t = \big(M \xrightarrow{\eta} (FM)^\# \xrightarrow{l^\#} A^\#\big),$$
$${}^t x = \big(FM \xrightarrow{Fx} F(A^\#) \xrightarrow{\varepsilon} A\big) \longleftarrow (x : M \to A^\#).$$

Define three classes of morphisms in $\mathcal{C}$:

$$\begin{aligned}
\mathcal{W} &= \{f \in \mathrm{Mor}\,\mathcal{C} \mid \forall\, x \in S \;\; f^\#(x) \text{ is a quasi-isomorphism}\},\\
\mathcal{R}_f &= \{f \in \mathrm{Mor}\,\mathcal{C} \mid \forall\, x \in S \;\forall\, z \in \mathbb{Z} \;\; f^\#(x)^z \text{ is surjective}\},\\
\mathcal{L}_c &= {}^\perp\mathcal{R}_{tf} \text{ consists of maps } f \in \mathrm{Mor}\,\mathcal{C} \text{ with the left lifting property}\\
&\qquad \text{with respect to all morphisms from } \mathcal{R}_{tf} = \mathcal{W} \cap \mathcal{R}_f.
\end{aligned}$$

We are going to prove that they are weak equivalences, fibrations and cofibrations of a certain model structure on $\mathcal{C}$.

Let $M \in \mathrm{Ob}\,\mathbf{dg}^S$, $A \in \mathrm{Ob}\,\mathcal{C}$, $\alpha : M \to A^\# \in \mathbf{dg}^S$. Denote by $C = \mathrm{Cone}\,\alpha = (M[1] \oplus UA, d_{\mathrm{Cone}}) \in \mathrm{Ob}\,\mathbf{dg}^S$ the cone taken pointwise, that is, for any $x \in S$ the complex $C(x) = \mathrm{Cone}\big(\alpha(x) : M(x) \to (UA)(x)\big)$ is the usual cone. Denote by $\bar\imath = \mathrm{in}_2 : UA \to C$ the obvious embedding. Following Hinich [Hin97, Section 2.2.2] define an object $A\langle M, \alpha\rangle \in \mathrm{Ob}\,\mathcal{C}$ as the pushout

$$\begin{tikzcd}
FU(A) \arrow[rr, "\varepsilon"] \arrow[d, "F\bar\imath"'] & & A \arrow[d, "\bar\jmath"] \\
FC \arrow[rr, "g"] & \arrow[ur, phantom, "\ulcorner"] & A\langle M, \alpha\rangle
\end{tikzcd}$$

Introduce a functor $h_{A,\alpha} : \mathcal{C} \to \mathcal{S}et$:

$$h_{A,\alpha}(B) = \big\{(f,t) \in \mathcal{C}(A,B) \times \underline{\mathbf{dg}}^S(M, B^\#)^{-1} \mid (t)d \equiv t d_{B^\#} + d_M t = \big(M \xrightarrow{\alpha} A^\# \xrightarrow{f^\#} B^\#\big)\big\}.$$

2.1 Lemma. *The object $D = A\langle M, \alpha\rangle$ and the element $(\bar{\jmath}, \theta) \in h_{A,\alpha}(D)$ represent the functor $h_{A,\alpha}$, where*

$$\theta = \bigl(M \xrightarrow{\sigma} M[1] \xrightarrow{\mathrm{in}_1} C \xrightarrow{\eta} UFC \xrightarrow{Ug} UD\bigr).$$

That is, the natural in B transformation $\psi_B : \mathcal{C}(D, B) \to h_{A,\alpha}(B)$, $1_D \mapsto (\bar{\jmath}, \theta)$, is bijective.

Proof. The boundary of degree -1 map $h = \bigl(M \xrightarrow{\sigma} M[1] \xrightarrow{\mathrm{in}_1} C\bigr)$ is $(h)d = hd_C + d_M h = \alpha \cdot \bar{\imath}$. Therefore, $(\theta)d$ is the composition along the bottom path in the diagram

$$\begin{tikzcd}
M \arrow[r, "\alpha"] & UA \arrow[r, "\eta"] \arrow[d, "\bar{\imath}"'] \arrow[dr, phantom, "="] & UFUA \arrow[r, "U\varepsilon"] \arrow[d, "UF\bar{\imath}"'] \arrow[dr, phantom, "="] & UA \arrow[d, "U\bar{\jmath}"] \\
& C \arrow[r, "\eta"] & UFC \arrow[r, "Ug"] & UD
\end{tikzcd}$$

which equals to the top path, that is, to $\alpha \cdot U\bar{\jmath}$. Therefore, $(\bar{\jmath}, \theta) \in h_{A,\alpha}(D)$. By the Yoneda lemma the natural transformation ψ_B takes a morphism $k : D \to B$ of $\mathcal{C}$ to

$$h_{A,\alpha}(k)(\bar{\jmath}, \theta) = \bigl(A \xrightarrow{\bar{\jmath}} D \xrightarrow{k} B, M \xrightarrow{h} C \xrightarrow{\eta} (FC)^{\#} \xrightarrow{g^{\#}} D^{\#} \xrightarrow{k^{\#}} B^{\#}\bigr). \tag{2.1}$$

Let us prove injectivity of ψ_B. Let $k_1, k_2 : D \to B$ satisfy

$$(f_1, t_1) \equiv h_{A,\alpha}(k_1)(\bar{\jmath}, \theta) = h_{A,\alpha}(k_2)(\bar{\jmath}, \theta) \equiv (f_2, t_2).$$

Then

$$\bigl(M[1] \xrightarrow{\mathrm{in}_1} C \xrightarrow{\eta} (FC)^{\#} \xrightarrow{g^{\#}} D^{\#} \xrightarrow{k_p^{\#}} B^{\#}\bigr) = \sigma^{-1} t_p$$

does not depend on $p = 1, 2$. On the other summand of C we also have that

$$\bigl(A^{\#} \xrightarrow{\bar{\imath}} C \xrightarrow{\eta} (FC)^{\#} \xrightarrow{g^{\#}} D^{\#} \xrightarrow{k_p^{\#}} B^{\#}\bigr) = \bigl(A^{\#} \xrightarrow{\bar{\jmath}^{\#}} D^{\#} \xrightarrow{k_p^{\#}} B^{\#}\bigr) = f_p^{\#}$$

does not depend on $p = 1, 2$. Therefore,

$$l_p^t = \bigl(C \xrightarrow{\eta} (FC)^{\#} \xrightarrow{g^{\#}} D^{\#} \xrightarrow{k_p^{\#}} B^{\#}\bigr)$$

also does not depend on $p = 1, 2$. Their adjuncts $l_p = \bigl(FC \xrightarrow{g} D \xrightarrow{k_p} B\bigr)$ must not depend on p either. By assumption

$$\bigl(A \xrightarrow{\bar{\jmath}} D \xrightarrow{k_1} B\bigr) = f_1 = f_2 = \bigl(A \xrightarrow{\bar{\jmath}} D \xrightarrow{k_2} B\bigr).$$

The pushout property of D allows only one morphism $D \to B$ with such properties, hence, $k_1 = k_2$.

Let us prove surjectivity of ψ_B. Given an element $(f : A \to B, t : M \to B^\#) \in h_{A,\alpha}(B)$ we construct a degree 0 map $x : C \to B^\#$

$$x = \begin{pmatrix} M[1] \xrightarrow{\sigma^{-1}} M \xrightarrow{t} B^\# \\ A^\# \xrightarrow{f^\#} B^\# \end{pmatrix}.$$

One easily checks that x is a chain map, $x \in \mathbf{dg}^S$. Its adjunct is denoted

$$l = {}^t x = \big(FC \xrightarrow{Fx} F(B^\#) \xrightarrow{\varepsilon} B\big).$$

Since $\bar\imath \cdot x = f^\# : A^\# \to B^\#$, we have

$$F\bar\imath \cdot l = \big(F(A^\#) \xrightarrow{F(f^\#)} F(B^\#) \xrightarrow{\varepsilon} B\big) = \varepsilon \cdot f.$$

By definition of pushout D there exists a unique morphism $k : D \to B \in \mathcal{C}$ such that $f = \bar\jmath \cdot k$, $l = g \cdot k$. Hence,

$$x = l^t = \big(C \xrightarrow{\eta} (FC)^\# \xrightarrow{l^\#} B^\#\big),$$

$$t = \big(M \xrightarrow{\sigma} M[1] \xrightarrow{\mathrm{in}_1} C \xrightarrow{x} B^\#\big) = \big(M \xrightarrow{\sigma} M[1] \xrightarrow{\mathrm{in}_1} C \xrightarrow{\eta} (FC)^\# \xrightarrow{g^\#} D^\# \xrightarrow{k^\#} B^\#\big).$$

Therefore, $\psi_B(k) = (f, t)$ and ψ_B is bijective. $\square$

2.2 Corollary. *The map $\big(M \xrightarrow{\alpha} A^\# \xrightarrow{\bar\jmath^\#} A\langle M, \alpha\rangle^\#\big) = (\theta)d$ is null-homotopic. If $d_M = 0$, then for any cycle $m \in ZM$ the cycle $m\alpha \in ZA^\#$ is taken by $\bar\jmath^\#$ to the boundary of the element $m\theta \in A\langle M, \alpha\rangle^\#$.*

Thus, when $F : \mathbf{dg}^S \to \mathcal{C}$ is the functor of constructing a free **dg**-algebra of some kind, the maps $\bar\jmath$ are interpreted as "adding variables to kill cycles".

The following statement is well-known.

2.3 Lemma. *Assume that $g : U \to V \in \mathsf{C}_\Bbbk$ is a surjective quasi-isomorphism. Then for any pair (u, v), $u \in U^{n+1}$, $v \in V^n$, such that $ud = 0$, $ug = vd$ there is an element $w \in U^n$ such that $wd = u$, $wg = v$.*

Proof. Vanishing of $H^{n+1}(g)[u] = [gu] = 0$ implies vanishing of the cohomology class $[u] = 0$. There is $y \in U^n$ such that $yd = u$. Denote $c = yg \in V^n$, then

$$cd = ygd = ydg = ug = vd.$$

Hence, $c - v$ is a cycle, and there is a cycle $z \in Z^n U$ such that $[zg] = [c - v]$. There is $e \in V^{n-1}$ such that $zg = c - v + ed$. The element e lifts to $x \in U^{n-1}$ such that $xg = e$. Thus,

$$yg = c = zg - xgd + v = (z - xd)g + v.$$

Therefore, $w = y - z + xd$ satisfies $wg = v$ and $wd = u$. $\square$

We say that M consists of free $\Bbbk$-modules if for any $x \in S$ the graded $\Bbbk$-module $M(x)$ is free – isomorphic to $\oplus_{a\in\mathbb{Z}} P^a \Bbbk[a]$ for some graded set P and $d_M = 0$.

2.4 Proposition. *Let M consist of free $\Bbbk$-modules, $d_M = 0$, $A \in \operatorname{Ob}\mathcal{C}$ and $\alpha : M \to A^\# \in \mathbf{dg}^S$. Then $\bar{\jmath} : A \to A\langle M, \alpha\rangle \in \mathcal{L}_c$.*

Proof. Let the image $y^\#$ of a morphism $y : U \to V \in \mathcal{C}$ be an epimorphism and a quasi-isomorphism. Let $u : A \to U \in \mathcal{C}$. Morphisms $v : A\langle M, \alpha\rangle \to V$, which make the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\bar{\jmath}"'] & U \arrow[d, two heads, "y", "\wr"'] \\ A\langle M, \alpha\rangle \arrow[r, "v"] \arrow[ur, dashed, "w"] & V \end{tikzcd} \tag{2.2}$$

commutative, are in bijection with elements $(A \xrightarrow{u} U \xrightarrow{y} V, M \xrightarrow{t} V^\#) \in h_{A,\alpha}(V)$. Thus,

$$(t)d = d_M t + t d_{V^\#} = \big(M \xrightarrow{\alpha} A^\# \xrightarrow{u^\#} U^\# \xrightarrow{y^\#} V^\#\big).$$

For some graded set $P = (P^a(s) \mid a \in \mathbb{Z}, s \in S)$, $P^a(s) \in \mathcal{S}\mathrm{et}$, we have $M = P\Bbbk = \big(\oplus_{a\in\mathbb{Z}} P^a(s)\Bbbk[a]\big)_{s\in S}$. Let us denote the chosen basis of M by $(e_p)_{p\in P^\bullet(\bullet)}$, $\deg e_p = \deg p$. For an arbitrary $p \in P^a(s)$ denote $n = a - 1$. We have a cycle $e_p\alpha u^\# \in Z^{n+1}(U^\#)$ and an element $e_p t \in (V^\#)^n$ such that $(e_p\alpha u^\#)y^\# = (e_p t)d_{V^\#}$. By Lemma 2.3 there is an element denoted $(e_p r) \in (U^\#)^n$ such that $e_p\alpha u^\# = (e_p r)d_{U^\#}$ and $e_p t = (e_p r)y^\#$. Choosing such $e_p r$ for all $p \in P^\bullet(\bullet)$ we get a map $r \in \underline{\mathbf{dg}}^S(M, U^\#)^{-1}$ such that the triangles commute

$$\begin{tikzcd} A^\# \arrow[r, "u^\#"] & U^\# \\ M \arrow[u, "\alpha"] \arrow[ur, "(r)d"'] & \end{tikzcd} \quad , \qquad \begin{tikzcd} & U^\# \arrow[d, "y^\#"] \\ M \arrow[ur, "r"] \arrow[r, "t"'] & V^\# \end{tikzcd} \quad .$$

Thus a pair $(u : A \to U, r : M \to U^\#) \in h_{A,\alpha}(U)$ determines a morphism $w : A\langle M, \alpha\rangle \to U \in \mathcal{C}$ by Lemma 2.1. Due to (2.1) the equation

$$u = \big(A \xrightarrow{\bar{\jmath}} A\langle M, \alpha\rangle \xrightarrow{w} U\big)$$

holds. Naturality of bijection ψ,

$$\begin{tikzcd} h_{A,\alpha}(U) \arrow[r, "\psi_U", "\sim"'] \arrow[d, "h_{A,\alpha}(U)"'] \arrow[dr, phantom, "="] & \mathcal{C}(A\langle M, \alpha\rangle, U) \arrow[d, "\mathcal{C}(1,y)"] \\ h_{A,\alpha}(V) \arrow[r, "\psi_V", "\sim"'] & \mathcal{C}(A\langle M, \alpha\rangle, V) \end{tikzcd}$$

applied to the pair (u, r) gives

$$\begin{tikzcd} (u : A \to U, r : M \to U^{\#}) \arrow[r, mapsto] \arrow[d, mapsto, "{(-\cdot y, -\cdot y^{\#})}"'] \arrow[dr, phantom, "="] & w \arrow[d, mapsto, "-\cdot y"] \\ (uy : A \to V, ry^{\#} : M \to V^{\#}) = (\bar{\jmath}v, t) \arrow[r, mapsto] & v = wy. \end{tikzcd}$$

This gives another equation

$$v = \big(A\langle M, \alpha\rangle \xrightarrow{w} U \xrightarrow{y} V\big)$$

and w is the sought diagonal filler for (2.2). □

If M consists of free $\Bbbk$-modules (and $d_M = 0$), then $\bar{\jmath} : A \to A\langle M, \alpha\rangle$ is a cofibration. It might be called an *elementary standard cofibration*. If

$$A \to A_1 \to A_2 \to \cdots$$

is a sequence of elementary standard cofibrations, B is a colimit of this diagram, then the "infinite composition" map $A \to B$ is a cofibration called a *standard cofibration* [Hin97, Section 2.2.3].

2.5 Lemma. *Let $\alpha \sim \alpha' : M \to A^{\#}$. Then there is a natural in B bijection $h_{A,\alpha}(B) \simeq h_{A,\alpha'}(B)$. Hence, there is an isomorphism k of representing objects, which is the last arrow in the equation which holds in $\mathcal{C}$:*

$$\bar{\jmath}' = \big(A \xrightarrow{\bar{\jmath}} A\langle M, \alpha\rangle \xrightarrow[\sim]{k} A\langle M, \alpha'\rangle\big).$$

Proof. Let $h \in \underline{\mathbf{dg}}^S(M, A^{\#})^{-1}$ be a homotopy, $\alpha - \alpha' = hd + dh : M \to A^{\#}$. Then we have well defined maps

$$\begin{array}{c} h_{A,\alpha}(B) = \big\{(f : A \to B, t : M \to B^{\#}) \mid (t)d = \alpha f^{\#}\big\} \\ \begin{array}{cc} (f, t) & (f, q + hf^{\#}) \\ \big\downarrow & \big\uparrow \\ (f, t - hf^{\#}) & (f, q) \end{array} \\ h_{A,\alpha'}(B) = \big\{(f : A \to B, q : M \to B^{\#}) \mid (q)d = \alpha' f^{\#}\big\} \end{array}$$

since

$$(t - hf^{\#})d = \alpha f^{\#} - (\alpha - \alpha')f^{\#} = \alpha' f^{\#},$$
$$(q + hf^{\#})d = \alpha' f^{\#} + (\alpha - \alpha')f^{\#} = \alpha f^{\#}.$$

These maps are mutually inverse and natural in B.

Take $B = A\langle M, \alpha' \rangle$. There is a commutative square of bijections

$$\begin{tikzcd}
\mathcal{C}(A\langle M, \alpha' \rangle, A\langle M, \alpha' \rangle) \arrow[r, "\mathcal{C}(k,1)", "\sim"'] \arrow[d, "\psi"', "\wr"] & \mathcal{C}(A\langle M, \alpha \rangle, A\langle M, \alpha' \rangle) \arrow[d, "\psi", "\wr"'] \\
h_{A,\alpha'}(A\langle M, \alpha' \rangle) \arrow[r, "\sim"] & h_{A,\alpha}(A\langle M, \alpha' \rangle)
\end{tikzcd}$$

which gives the equation

$$\begin{tikzcd}
1_B \arrow[rr, mapsto] \arrow[d, mapsto] & & k \arrow[d, mapsto] \\
(\bar\jmath', t') \arrow[r, mapsto] & (\bar\jmath', t' + h\bar\jmath'^{\#}) \arrow[r, equal] & (\bar\jmath k, tk^{\#}).
\end{tikzcd}$$

In particular, $\bar\jmath' = \bar\jmath k$. $\square$

2.6 Remark. Consider a diagram $\alpha' = \left(M' \xrightarrow{\beta} M'' \xrightarrow{\alpha''} A^{\#}\right)$ in $\mathbf{dg}^S$. These morphisms lead to natural transformation $h_{A,\alpha''}(B) \to h_{A,\alpha'}(B)$, $(f, t) \mapsto (f, \beta \cdot t)$, or equivalently $\mathcal{C}(A\langle M'', \alpha'' \rangle, B) \to \mathcal{C}(A\langle M', \alpha' \rangle, B)$, which comes from a unique morphism $A\langle \beta \rangle : A\langle M', \beta \cdot \alpha'' \rangle \to A\langle M'', \alpha'' \rangle \in \mathcal{C}$. It can be found from the diagram

$$\begin{tikzcd}
 & F(A^{\#}) \arrow[r, "\varepsilon"] \arrow[d, "F\bar\imath'"] \arrow[dl, "F\bar\imath''"'] & A \arrow[d, "\bar\jmath'"'] \arrow[dr, "\bar\jmath''"] & \\
 & F(C') \arrow[r, "g'"] \arrow[dl, "F\gamma"] & A\langle M', \alpha' \rangle \arrow[dr, "A\langle \beta \rangle"'] \arrow[ul, phantom, "\ulcorner", very near start] & \\
F(C'') \arrow[rrr, "g''"] & & & A\langle M'', \alpha'' \rangle
\end{tikzcd} \tag{2.3}$$

where $\gamma = \mathrm{Cone}(\beta, 1) : C' \to C''$ is the morphism of cones, induced by β.

In fact, put $B = A\langle M'', \alpha'' \rangle$. The unit morphism 1_B corresponds to $(\bar\jmath'', \theta'') \in h_{A,\alpha''}(B)$ which is taken to $(\bar\jmath'', \beta \cdot \theta'') \in h_{A,\alpha'}(B)$. The latter element has to coincide with $(\bar\jmath' \cdot A\langle \beta \rangle, \theta' \cdot A\langle \beta \rangle^{\#})$. The equation $\bar\jmath'' = \bar\jmath' \cdot A\langle \beta \rangle$ is the right triangle of (2.3). The equation $\beta \cdot \theta'' = \theta' \cdot A\langle \beta \rangle^{\#}$ can be written as the exterior of

$$\begin{tikzcd}
M' \arrow[r, "\sigma"] \arrow[d, "\beta"'] & M'[1] \arrow[r, "\mathrm{in}_1"] & C' \arrow[r, "g'^t"] \arrow[d, "\gamma"'] & D'^{\#} \arrow[d, "A\langle \beta \rangle^{\#}"] \\
M'' \arrow[r, "\sigma"] & M''[1] \arrow[r, "\mathrm{in}_1"] & C'' \arrow[r, "g''^t"] & D''^{\#}
\end{tikzcd}$$

The mentioned right triangle implies commutativity of the exterior of

$$\begin{tikzcd}
A^{\#} \arrow[r, "\bar\imath'"] \arrow[d, equal] & C' \arrow[r, "g'^t"] \arrow[d, "\gamma"'] & D'^{\#} \arrow[d, "A\langle \beta \rangle^{\#}"] \\
A^{\#} \arrow[r, "\bar\imath''"] & C'' \arrow[r, "g''^t"] & D''^{\#}
\end{tikzcd}$$

This fact jointly with the previous implies commutativity of the right square, which is equivalent to lower trapezia in (2.3).

In particular, for $0 = \bigl(0 \xrightarrow{0} M \xrightarrow{\alpha} A^{\#}\bigr)$ we have $\bar{\imath}' = \mathrm{id} : A^{\#} \to C'$, $\bar{\jmath}' = \mathrm{id} : A \to A\langle 0,0\rangle$, $\bar{\jmath}'' = \bar{\jmath} = A\langle 0\rangle : A = A\langle 0,0\rangle \to A\langle M,\alpha\rangle$.

2.7 Remark. For $0 : M \to A^{\#}$ we have that $A\langle M,0\rangle \simeq F(M[1]) \sqcup A$ and $\bar{\jmath} = \mathrm{in}_2$ is the canonical embedding. In fact, $C = M[1] \oplus A^{\#}$ is the direct sum of complexes and $A\langle M,0\rangle$ is found from the following diagram

$$\begin{tikzcd}
F(A^{\#}) \arrow[rr, "\varepsilon"] \arrow[d, "F(\mathrm{in}_2)"'] \arrow[dr, "\mathrm{in}_2"] & & A \arrow[d, "\mathrm{in}_2"] \\
F(M[1] \sqcup A^{\#}) \arrow[r, "\sim"] & F(M[1]) \sqcup F(A^{\#}) \arrow[r, "1 \sqcup \varepsilon"] & F(M[1]) \sqcup A = A\langle M,0\rangle
\end{tikzcd}$$

2.8 Example. Let $N \in \operatorname{Ob} \mathbf{dg}^S$. Take FN for A and $\eta : N \to (FN)^{\#}$ for α. We claim that we can take $F(\operatorname{Cone} 1_N)$ for $(FN)\langle N,\eta\rangle$. In fact,

$$\begin{aligned}
h_{FN,\eta}(B) &= \bigl\{(f : FN \to B, t : N \to B^{\#}) \mid (t)d = \eta \cdot f^{\#}\bigr\} = \bigl\{(f,t) \mid (t)d = f^t\bigr\} \\
&= \bigl\{(f,t) \mid f = {}^t((t)d)\bigr\} = \bigl\{t \in \underline{\mathbf{dg}}^S(N,B^{\#})^{-1}\bigr\} \simeq \underline{\mathbf{dg}}^S(N[1],B^{\#})^0 \\
&\overset{(!)}{\simeq} \mathbf{dg}^S\bigl((N[1] \oplus N, d_{\operatorname{Cone} 1_N}), B^{\#}\bigr) = \mathbf{dg}^S(\operatorname{Cone} 1_N, B^{\#}) \simeq \mathcal{C}(F(\operatorname{Cone} 1_N), B).
\end{aligned}$$

Bijection (!) is left to the reader as an exercise.

2.9 Proposition. *Let $N = P\Bbbk \in \mathbf{dg}^S$ consist of free $\Bbbk$-modules, $d_N = 0$, and $M = \operatorname{Cone} 1_{N[-1]} = (N \oplus N[-1], d_{\mathrm{Cone}})$. Then for any morphism $\alpha : M \to UA \in \mathbf{dg}^S$ the morphism $\bar{\jmath} : A \to A\langle M,\alpha\rangle$ is a standard cofibration, composition of two elementary standard cofibrations.*

Proof. The complex M is contractible, hence, $\alpha \sim 0 = \alpha' : M \to A^{\#}$. Applying Lemma 2.5 we find that $\bar{\jmath} = \bigl(A \xrightarrow{A\langle 0\rangle} A\langle M,0\rangle \xrightarrow{\sim} A\langle M,\alpha\rangle\bigr)$, thus it suffices to prove the claim for $\alpha = 0$.

The embedding $\mathrm{in}_2 : N[-1] \to M$ induces a diagram

$$\begin{tikzcd}
A \arrow[r, "\bar{\jmath}'"] \arrow[dr, "\bar{\jmath}''"'] & A\langle N[-1],0\rangle \arrow[r, equal] \arrow[d, "A\langle \mathrm{in}_2\rangle"] & A\langle N[-1],0\rangle\langle 0,0\rangle \arrow[d, "A\langle N[-1],0\rangle\langle 0\rangle"] \\
& A\langle M,0\rangle \arrow[r, "\sim"] & A\langle N[-1],0\rangle\langle N,\eta\rangle
\end{tikzcd}$$

Commutativity of the triangle is contained in diagram (2.3). Commutativity of the square is implied by Remark 2.7, which gives $A\langle N[-1],0\rangle = FN \sqcup A$, and by the equation

$$\begin{tikzcd}
FN \arrow[r, equal] \arrow[d, "F(\mathrm{in}_2)"'] \arrow[dr, phantom, "="] & (FN)\langle 0,0\rangle \arrow[d, "(FN)\langle 0\rangle"] \\
F(\operatorname{Cone} 1_N) \arrow[r, equal] & (FN)\langle N,\eta\rangle.
\end{tikzcd}$$

The latter equation follows from Example 2.8. Let us take $B = F(\operatorname{Cone} 1_N)$ in it and find the element of $h_{FN,\eta}(F(\operatorname{Cone} 1_N))$, which goes into 1_B under the sequence of bijections considered in the example. Moving backwards we find elements

$$
\begin{aligned}
1_B \mapsto \langle \eta : \operatorname{Cone} 1_N \to (F \operatorname{Cone} 1_N)^{\#} \rangle \mapsto \Big\langle N[1] \xrightarrow{\mathrm{in}_1} \operatorname{Cone} 1_N \xrightarrow{\eta} (F \operatorname{Cone} 1_N)^{\#} \Big\rangle \\
\mapsto t = \Big\langle N \xrightarrow{\sigma} N[1] \xrightarrow{\mathrm{in}_1} \operatorname{Cone} 1_N \xrightarrow{\eta} (F \operatorname{Cone} 1_N)^{\#} \Big\rangle \in \underline{\mathbf{dg}}^S(N, B^{\#})^{-1}.
\end{aligned}
$$

Computation in the proof of Lemma 2.1 give

$$
(t).d = \Big\langle N \xrightarrow{\mathrm{in}_2} \operatorname{Cone} 1_N \xrightarrow{\eta} (F \operatorname{Cone} 1_N)^{\#} \Big\rangle = \Big\langle N \xrightarrow{\eta} (FN)^{\#} \xrightarrow{(F \,\mathrm{in}_2)^{\#}} (F \operatorname{Cone} 1_N)^{\#} \Big\rangle,
$$

hence t comes from the pair $(F(\mathrm{in}_2), t) \in h_{FN,\eta}(F(\operatorname{Cone} 1_N))$. Thus, $\bar{\jmath}'' : A \to A\langle M, 0\rangle$ is a composition of two elementary standard cofibrations and a standard cofibration itself. □

2.10 Proposition. *Let $r : A \to Y \in \mathcal{C}$. Denote by*

$$
\begin{aligned}
N &= Z \operatorname{Cone}(r^{\#}[-1] : A^{\#}[-1] \to Y^{\#}[-1]) \\
&= \big\{ (u, y\sigma^{-1}) \in A^{\#} \times Y^{\#}[-1] \mid ud = 0,\ ur^{\#} - yd_{Y^{\#}} = 0 \big\}
\end{aligned}
$$

the differential graded $\Bbbk$-submodule of cycles of $\operatorname{Cone}(r^{\#}[-1])$, $d_N = 0$. *Denote by* $\mathrm{pr}_1 : N \to A^{\#} \in \mathbf{dg}^S$ *(resp.* $\mathrm{pr}_2 : N \to Y^{\#}[-1] \in \mathbf{gr}^S$*) the map $(u, y\sigma^{-1}) \mapsto u$ (resp. $(u, y\sigma^{-1}) \mapsto y\sigma^{-1}$). Define $D = A\langle N, \mathrm{pr}_1\rangle$. Then*

$$
\big(r : A \to Y, t = (N \xrightarrow{\mathrm{pr}_2} Y^{\#}[-1] \xrightarrow{\sigma} Y^{\#}) \big)
$$

is an element of $h_{A,\mathrm{pr}_1}(Y)$. The corresponding morphism $q : D \to Y$ satisfies $r = \big(A \xrightarrow{\bar{\jmath}} A\langle N, \mathrm{pr}_1\rangle \xrightarrow{q} Y \big)$. The composition

$$
\beta = \Big\langle N \hookrightarrow \operatorname{Cone}(r^{\#}[-1]) \xrightarrow{\operatorname{Cone}(\bar{\jmath}^{\#}[-1],1)} \operatorname{Cone}(q^{\#}[-1]) \Big\rangle, \quad \operatorname{Cone}(\bar{\jmath}^{\#}[-1], 1) = \begin{pmatrix} \bar{\jmath}^{\#} & 0 \\ 0 & 1 \end{pmatrix},
$$

is null-homotopic, $\beta = (\theta, 0).d = (\theta, 0) \cdot d_{\operatorname{Cone}(q^{\#}[-1])}$, thus, all cycles of $\operatorname{Cone}(r^{\#}[-1])$ *are taken by* $\operatorname{Cone}(\bar{\jmath}^{\#}[-1], 1_{Y^{\#}[-1]})$ *to boundaries in* $\operatorname{Cone}(q^{\#}[-1])$.

Proof. Let us show that $(r, t) \in h_{A,\mathrm{pr}_1}(Y)$. In fact, the diagram

$$
\begin{tikzcd}
N \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y^{\#}[-1] \arrow[r, "\sigma"] & Y^{\#} \arrow[d, "d_{Y^{\#}}"] \\
A^{\#} \arrow[rr, "r^{\#}"] & & Y^{\#}
\end{tikzcd}
$$

commutes as the computation shows

$$
\begin{tikzcd}
(u, y\sigma^{-1}) \arrow[r, mapsto] \arrow[d, mapsto] & y\sigma^{-1} \arrow[r, mapsto] & y \arrow[d, mapsto] \\
u \arrow[rr, mapsto] & & ur^{\#} = yd_{Y^{\#}}
\end{tikzcd}
$$

The corresponding morphism $q : D \to Y$ satisfies $(r,t) = \big(\bar{\jmath}\cdot q, N \xrightarrow{\theta} D^\# \xrightarrow{q^\#} Y^\#\big)$ by (2.1).

One can easily check that cones are related by the chain map

$$\mathrm{Cone}(\bar{\jmath}^\#[-1], 1_{Y^\#[-1]}) = \begin{pmatrix} \bar{\jmath}^\# & 0 \\ 0 & 1_{Y^\#[-1]} \end{pmatrix} : \mathrm{Cone}((\bar{\jmath}^\# q^\#)[-1]) \to \mathrm{Cone}(q^\#[-1]).$$

The composition β takes $(u, y\sigma^{-1}) \in N$ to $(u\bar{\jmath}^\#, y\sigma^{-1}) \in \mathrm{Cone}(q^\#[-1])$. Since $d_N = 0$ the map

$$(\theta,0).d = (\theta,0)\begin{pmatrix} d_{D^\#} & q^\#\sigma^{-1} \\ 0 & d_{Y^\#[-1]} \end{pmatrix} = (\mathrm{pr}_1 \cdot \bar{\jmath}^\#, \theta q^\# \sigma^{-1}) = (\mathrm{pr}_1 \cdot \bar{\jmath}^\#, t\sigma^{-1}) = (\mathrm{pr}_1 \cdot \bar{\jmath}^\#, \mathrm{pr}_2)$$

takes $(u, y\sigma^{-1})$ to the same $(u\bar{\jmath}^\#, y\sigma^{-1})$ as β. $\square$

Assume hypotheses of Theorem 1.2 hold.

2.11 Proposition. *Let $N = P\Bbbk \in \mathbf{dg}^S$ consist of free $\Bbbk$-modules, $d_N = 0$, and $M = \mathrm{Cone}\, 1_{N[-1]}$. Then for all $\alpha : M \to A^\# \in \mathbf{dg}^S$ the morphism $\bar{\jmath} : A \to A\langle M, \alpha\rangle$ is in $\mathcal{W}$.*

Proof. The complex M is contractible, hence, it suffices to consider $\alpha = 0$. Consider the directed set of finite graded subsets $Q \subset P$ (that is, the set $\coprod_{c\in\mathbb{Z}}^{x\in S} Q^c(x)$ is finite). We have

$$\begin{aligned} M[1] = P\mathbb{K}[1] &= \bigoplus_{c\in\mathbb{Z}}^{x\in S} P^c(x)\mathbb{K}_x[c+1] = \operatorname*{colim}_{Q\subset P} \coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} \mathbb{K}_x[c+1], \\ \bar{\jmath}^\# = \mathrm{in}_2^\# &= \big\langle A^\# \to (F(M[1]) \textstyle\coprod A)^\# \big\rangle \\ &= \Big\langle A^\# \to \Big(\operatorname*{colim}_{Q\subset P}\big(\coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} F(\mathbb{K}_x[c+1])\big) \textstyle\coprod A\Big)^\# \Big\rangle \\ &= \Big\langle A^\# \to \operatorname*{colim}_{Q\subset P}\Big(\big(\coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} F(\mathbb{K}_x[c+1])\big) \textstyle\coprod A\Big)^\# \Big\rangle. \end{aligned}$$

For any finite Q the map $\mathrm{in}_2^\# : A^\# \to \big(\big(\coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} F(\mathbb{K}_x[c+1])\big) \coprod A\big)^\#$ is a quasi-isomorphism as a finite composition of quasi-isomorphisms. Thus its cone is acyclic. Therefore, the cone

$$\begin{aligned} &\mathrm{Cone}\Big\langle \bar{\jmath}^\# : A^\# \to \operatorname*{colim}_{Q\subset P}\Big(\big(\coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} F(\mathbb{K}_x[c+1])\big) \textstyle\coprod A\Big)^\# \Big\rangle \\ &\qquad\simeq \operatorname*{colim}_{Q\subset P} \mathrm{Cone}\Big\langle A^\# \to \Big(\big(\coprod_{x\in S,\, c\in\mathbb{Z}}^{q\in Q^c(x)} F(\mathbb{K}_x[c+1])\big) \textstyle\coprod A\Big)^\# \Big\rangle \end{aligned}$$

is acyclic and $\bar{\jmath}^\#$ is a quasi-isomorphism. $\square$

To sum up Propositions 2.9 and 2.11 assume that $N \in \operatorname{Ob}\mathbf{dg}^S$ consists of free $\Bbbk$-modules, $d_N = 0$, and $M = \operatorname{Cone} 1_{N[-1]} = (N \oplus N[-1], d_{\mathrm{Cone}})$. Then for any morphism $\alpha : M \to UA \in \mathbf{dg}^S$ the morphism $\bar{\jmath} : A \to A\langle M, \alpha\rangle$ is a trivial cofibration in $\mathcal{C}$ and a standard cofibration, composition of two elementary standard cofibrations. It is called a *standard trivial cofibration*.

Proof of Theorem 1.2. (MC1) (Co)completeness of $\mathcal{C}$ is assumed. Axioms (MC2) (three-for-two for $\mathcal{W}$) and (MC3) (closedness of $\mathcal{L}_c$, $\mathcal{W}$, $\mathcal{R}_f$ with respect to retracts) are obvious. The class $\mathcal{L}_c$ is ${}^{\perp}(\mathcal{W} \cap \mathcal{R}_f)$ by definition.

(MC5)(ii) *Functorial factorization into a trivial cofibration and a fibration.* Let $f : X \to Y \in \mathcal{C}$. Denote $N = Y^{\#}\Bbbk$, $M[1] = \operatorname{Cone} 1_{N[-1]} = (N \oplus N[-1], d_{\mathrm{Cone}}) \simeq Y^{\#}\mathbb{K}[-1]$. The $\Bbbk$-linear degree 0 map $N \to Y^{\#}$, $e_y \mapsto y$, extends in a unique way to a degreewise surjection $\pi^t_Y : M[1] \to Y^{\#} \in \mathbf{dg}^S$, which determines a morphism $\pi_Y : F(M[1]) \to Y \in \mathcal{C}$. Combining it with the previous we get a morphism $\pi_Y \cup f : F(M[1]) \coprod X \to Y \in \mathcal{C}$. Since $\pi^t_Y = \big\langle M[1] \xrightarrow{\eta} (F(M[1]))^{\#} \xrightarrow{\pi_Y^{\#}} Y^{\#}\big\rangle$ is a surjection, the map $\pi_Y^{\#} = \big\langle (F(M[1]))^{\#} \xrightarrow{\mathrm{in}_1^{\#}} (F(M[1]) \coprod X)^{\#} \xrightarrow{(\pi_Y \cup f)^{\#}} Y^{\#}\big\rangle$ is a surjection as well. Therefore, $(\pi_Y \cup f)^{\#}$ is a surjection and $\pi_Y \cup f \in \mathcal{R}_f$. The decomposition

$$f = \big(X \xrightarrow{\bar{\jmath}} X\langle M, 0\rangle = F(M[1]) \textstyle\coprod X \xrightarrow{(\pi_Y \cup f)^{\#}} Y\big)$$

into a trivial cofibration and a fibration is functorial in f.

(MC5)(i) *Functorial factorization into a cofibration and a trivial fibration.* Let us construct inductively the following diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, equal] & D_0 \arrow[r, "h_0"] \arrow[drr, "f=q_0"'] & D_1 \arrow[r, "h_1"] \arrow[dr, "q_1"] & D_2 \arrow[r, "h_2"] \arrow[d, "q_2"] & \dots \\ & & & Y & \end{tikzcd} \qquad \dots \tag{2.4}$$

so that all h_i were cofibrations. Given q_n for $n \geqslant 0$ denote

$$\begin{aligned} N_n &= Z \operatorname{Cone}(q_n^{\#}[-1] : D_n^{\#}[-1] \to Y^{\#}[-1]) \\ &= \big\{(u, y\sigma^{-1}) \in D_n^{\#} \times Y^{\#}[-1] \mid ud = 0,\, uq_n^{\#} - yd_{Y^{\#}} = 0\big\} \end{aligned}$$

as in Proposition 2.10. Being a subset of cycles N_n is a graded $\Bbbk$-submodule with $d_{N_n} = 0$. Viewing N_n as a graded set introduce a graded $\Bbbk$-module $M_n = N_n\Bbbk$, $d_{M_n} = 0$, with the projection $p_n : M_n \longrightarrow\!\!\triangleright\, N_n \in \mathbf{dg}^S$, $e_v \mapsto v$ for all $v \in N_n^{\bullet}(\bullet)$. Let us denote $\alpha_n = \big(M_n \xrightarrow{p_n} N_n \xrightarrow{\mathrm{pr}_1} D_n^{\#}\big) \in \mathbf{dg}^S$. Choose $D_{n+1} = D_n\langle M_n, \alpha_n\rangle$, then $h_n = D_n\langle 0\rangle : D_n \to D_{n+1}$ is a cofibration. Proposition 2.10 and Remark 2.6 imply that $(q_n : D_n \to Y, t_n = \big(M_n \xrightarrow{p_n} N_n \xrightarrow{\mathrm{pr}_2} Y^{\#}[-1] \xrightarrow{\sigma} Y^{\#}\big)$ is an element of $h_{D_n,\alpha_n}(Y)$. A morphism

$q_{n+1} : D_{n+1} = D_n\langle M_n, \alpha_n\rangle \to Y \in \mathcal{C}$ corresponds to the pair (q_n, t_n) such that $q_n = \big(D_n \xrightarrow{h_n} D_{n+1} \xrightarrow{q_{n+1}} Y\big)$ in $\mathcal{C}$, which gives the required diagram.

Let us prove that $q_2^\# : D_2^\# \to Y^\#$ is surjective in all degrees. Let $y \in Y^{\#\bullet}(\bullet)$. Then $(0, yd\sigma^{-1}) \in N_0$, $e_{(0,yd\sigma^{-1})} \in M_0$, $e_{(0,yd\sigma^{-1})}\theta_0 \in D_1^\#$. The equation $\theta_0 q_1^\# = t_0 = p_0 \cdot \mathrm{pr}_2 \cdot\sigma : M_0 \to Y^\#$ implies that

$$e_{(0,yd\sigma^{-1})}\theta_0 q_1^\# - yd_{Y^\#} = (0, yd\sigma^{-1})\,\mathrm{pr}_2\,\sigma - yd = 0.$$

Furthermore,

$$e_{(0,yd\sigma^{-1})}\theta_0 d_{D_1^\#} = e_{(0,yd\sigma^{-1})}.(\theta)d = e_{(0,yd\sigma^{-1})}\alpha_0\bar{\imath}_0\eta g_0^\# = (0, yd\sigma^{-1})\,\mathrm{pr}_1\,\alpha_0\bar{\imath}_0\eta g_0^\# = 0.$$

Thus, $(e_{(0,yd\sigma^{-1})}\theta_0, y\sigma^{-1}) \in N_1$. Therefore the map $\mathrm{pr}_2 \cdot\sigma : N_1 \to Y^\#$ is surjective in each degree. Hence, the map $t_1 = \big(M_1 \xrightarrow{p_1} N_1 \xrightarrow{\mathrm{pr}_2} Y^\#[-1] \xrightarrow{\sigma} Y^\#\big)$ is surjective as well. Since $t_1 = \big(M_1 \xrightarrow{\theta_1} D_2^\# \xrightarrow{q_2^\#} Y^\#\big)$, it follows that $q_2^\#$ is surjective in each degree. Consequently $q_n^\# : D_n^\# \to Y^\#$ is surjective for all $n \geqslant 2$, and the induced map $q^\# : D^\# \to Y^\#$ is surjective as well, where

$$q = \operatorname*{colim}_{n\in\mathbb{N}} q_n : D = \operatorname*{colim}_{n\in\mathbb{N}} D_n \to Y.$$

Diagram (2.4) induces also diagram of cones

$$\mathrm{Cone}\, q_0^\# \xrightarrow{\mathrm{Cone}(h_0^\#,1)} \mathrm{Cone}\, q_1^\# \xrightarrow{\mathrm{Cone}(h_1^\#,1)} \mathrm{Cone}\, q_2^\# \to \cdots \to \mathrm{Cone}\, q^\# = \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Cone}\, q_n^\#.$$

It follows from Proposition 2.10 that the submodule of cycles $Z\,\mathrm{Cone}\, q_n^\#$ is taken by $\mathrm{Cone}(h_n^\#, 1)$ to the submodule of boundaries $B\,\mathrm{Cone}\, q_{n+1}^\#$. Thus the colimit of cones $\mathrm{Cone}\, q^\#$ is acyclic. Therefore, $q^\#$ is a quasi-isomorphism. We have decomposed a morphism $f \in \mathcal{C}$ into a standard cofibration i and a trivial fibration q: $f = \big(X \xrightarrow{i} D \xrightarrow{q} Y\big)$.

(MC4)(ii). Let us prove that a standard trivial cofibration $\bar{\jmath} : X \to X\langle M, 0\rangle$ is in ${}^\perp\mathcal{R}_f$. Here $M[1] = \mathrm{Cone}\, 1_{N[-1]}$ and N consists of free $\Bbbk$-modules. We have $X\langle M, 0\rangle = F(M[1]) \sqcup X$. A square

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\bar{\jmath}"'] & A \arrow[d, "g"] \\ X\langle M, 0\rangle \arrow[r, "b"] \arrow[ur, dashed, "c"] & B \end{tikzcd}$$

commutes iff $b = l \cup ag : F(M[1]) \sqcup X \to B$. The adjunction takes l to $l^t : M[1] \to B^\# \in \mathbf{dg}^S$. There is a commutative diagram in $\mathcal{S}et$

$$\begin{tikzcd} \mathbf{dg}^S(M[1], A^\#) \arrow[r, "\sim"] \arrow[d, "\mathbf{dg}^S(1,g^\#)"'] & \underline{\mathbf{dg}}^S(N, A^\#)^0 \arrow[d, "\underline{\mathbf{dg}}^S(1,g^\#)"] \\ \mathbf{dg}^S(M[1], B^\#) \arrow[r, "\sim"] & \underline{\mathbf{dg}}^S(N, B^\#)^0 \end{tikzcd} \tag{2.5}$$

Assume that $g \in \mathcal{R}_f$, that is, $g^\#$ is surjective in each degree. Since N consists of free $\Bbbk$-modules, the vertical maps are surjections. Thus, there is a chain map $r : M[1] \to A^\#$ such that $l^t = r \cdot g^\#$. Using adjunction we find that $l = \big(F(M[1]) \xrightarrow{{}^t r} A \xrightarrow{g} B\big)$. Then $c = {}^t r \cup a : F(M[1]) \sqcup X \to A$ is the sought diagonal filler.

Denote by J the class of all standard trivial cofibrations. Then the above reasoning turned backward shows that for $g \in J^\perp$ vertical arrows of (2.5) are always surjective which implies that $g \in \mathcal{R}_f$. Hence, $J^\perp = \mathcal{R}_f$.

Consider an arbitrary morphism $f : X \to Y \in \mathcal{C}$. Accordingly to proven (MC5)(ii) there is a decomposition

$$\begin{tikzcd} X \arrow[r, "\bar{\jmath}"] \arrow[d, "f"'] & Z = X\langle M, 0\rangle \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

into a standard trivial cofibration $\bar{\jmath}$ and $p \in \mathcal{R}_f$. If $f \in \mathcal{W} \cap \mathcal{L}_c$, then $p \in \mathcal{W} \cap \mathcal{R}_f$. By definition of $\mathcal{L}_c = {}^\perp(\mathcal{W} \cap \mathcal{R}_f)$ there is a morphism w such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\bar{\jmath}"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "w"] & Y \end{tikzcd} \iff \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "\bar{\jmath}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "w"] & Z \arrow[r, "p"] & Y \end{tikzcd} \tag{2.6}$$

and $w \cdot p = 1_Y$. That is, f is a retract of $\bar{\jmath}$. Hence, $(\mathcal{W} \cap \mathcal{R}_f)^\perp = J^\perp = \mathcal{R}_f$. □

2.12 Remark. It is shown in the proof that any trivial cofibration f is a retract of a standard trivial cofibration $\bar{\jmath}$ of type (2.6), *cf.* [Hin97, Remark 2.2.5]. Similarly, any cofibration f is a retract of a standard cofibration $\bar{\jmath}$ of type (2.6). The model structure of $\mathcal{C}$ is cofibrantly generated by the classes of elementary cofibrations and of standard trivial cofibrations.

References

[BM03] Clemens Berger and Ieke Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831, `arXiv:math.AT/0206094`.

[CH02] J. Daniel Christensen and Mark Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 2, 261–293, `arXiv:math.KT/0011216`.

[Har10] John Edward Harper, *Homotopy theory of modules over operads and non-Σ operads in monoidal model categories*, J. Pure Appl. Algebra **214** (2010), no. 8, 1407–1434, `arXiv:0801.0191`.

[Hin97] Vladimir Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), no. 10, 3291–3323, `arXiv:q-alg/9702015`.

[Lyu11] Volodymyr Lyubashenko, *Homotopy unital A_∞-algebras*, J. Algebra **329** (2011), no. 1, 190–212, `http://dx.doi.org/10.1016/j.jalgebra.2010.02.009` Special Issue Celebrating the 60th Birthday of Corrado De Concini `arXiv:1205.6058`.

[Lyu12] Volodymyr Lyubashenko, *A_∞-morphisms with several entries*, may 2012, `arXiv:1205.6072`.

[Mur11] Fernando Muro, *Homotopy theory of nonsymmetric operads*, Algebr. Geom. Topol. **11** (2011), no. 3, 1541–1599, `arXiv:1101.1634` `http://dx.doi.org/10.2140/agt.2011.11.1541`.

[Spi01] Markus Spitzweck, *Operads, algebras and modules in general model categories*, jan 2001, `arXiv:math/0101102`.

[SS00] Stefan Schwede and Brooke Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511, `arXiv:math.AT/9801082`.

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